(52) Examples for some simple codes:

Ex: Single parity check code

$$G = \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & & & & 1 & 1 \end{array}\right]_{k \times n}$$

$$\underset{1\times n}{\bar{c}} = \underset{1\times k}{\bar{d}}\,\underset{k\times n}{G} \qquad \bar{d} = (d_1 \ldots d_k)$$

$$\bar{c} = (d_1 \ldots d_k \;\; d_1 + d_2 + \ldots d_k)$$

Ex: Repetition Code

$$G = (1\,1 \ldots 1)_{1\times n}$$

$$\bar{c} = d\,G \qquad \bar{c} = (d\,d \ldots d)_{1\times n}$$

$$d \rightarrow \text{bit}$$

<u>Error Correction:</u>

Syndrome:

Let G be the generator matrix of the Code C

Let H " " " " " the dual code

or parity check matrix of C

if $G = [I \;\; P] \longrightarrow$ then $H = [-P^T \;\; I]$

Let $\bar{d}_i$ denote the dataword to be transmitted

For the transmission of $\bar{d}_i$ we first perform the encoding operation

$\bar{c}_i = \bar{d}_i G$

$\bar{d}_i \xrightarrow{\text{encode}} \bar{c}_i \xrightarrow{\text{transmission}} \bar{r}_i = \bar{c}_i + \bar{e}_i$

received word

$\bar{e}_i \rightarrow$ error word

If there is no error during transmission then $\bar{e}_i = 0$ In this case $\bar{r}_i = c_i$

we know that $\bar{c} H^T = 0$

Thus if $\bar{r} H^T = 0$ then we say that there is no error during transmission

but if $r H^T \neq 0$ then there is error during transmission.

$S = r H^T \rightarrow S$ is called syndrome for r

Code Rate:
$\bar{d} \rightarrow 1 \times k$
$\bar{c} \rightarrow 1 \times n$ $\quad \bar{c} = \bar{d} . G \quad G \rightarrow k \times n$

Code rate of C is $\frac{k}{n}$

$R_C = \frac{k}{n} \rightarrow$ code rate of C

C is generated by G

Error Detection & Correction using syndromes

Standard arrays:

$C \rightarrow$ code generated by $G_{k \times n}$

2^k codewords exist

let $M = 2^k$

$C = \{ \bar{c}_0, \bar{c}_1, \ldots \bar{c}_{M-1} \} \rightarrow$ Code or Codebook

$C(n,k) \rightarrow$ code rate is $\frac{k}{n}$

Standard array for $C(n,k)$ linear code is formed of

$M = 2^k$

$L = 2^{n-k}$

$$
2^{n-k} \text{ rows} \left\{
\begin{array}{cccc}
c_0 & c_1 & \cdots & c_{M-1} \\
w_1 & c_1 + w_1 & \cdots & c_{M-1} + w_1 \\
w_2 & c_1 + w_2 & \cdots & c_{M-1} + w_2 \\
\vdots & & & \vdots \\
w_{L-1} & c_1 + w_{L-1} & \cdots & c_{M-1} + w_{L-1}
\end{array}
\right.
$$

2^k column

Total number of words in standard array is $2^{n-k} \times 2^k = 2^n$

The rows of the standard array are called cosets and the first word in each coset is called the coset leader.

(55) For a t-error correcting code, a horizontal line is drawn in the array to separate rows whose coset leaders have weights less than or equal to t, from the rows whose coset leaders have weight greater than t.

Ex: Construct the standard array for the code with the generator matrix

$$G = \begin{bmatrix} 1 & 0 & | & 1 & 1 & 1 \\ 0 & 1 & | & 1 & 0 & 1 \end{bmatrix}_{2\times 5}$$

Sln: The code generated by G is

$$C = \{00000,\ 10111,\ 01101,\ 11010\}$$

$d_{min}(C) = 3 \rightarrow$ minimum distance of C

$\left\lfloor \frac{d_{min}-1}{2} \right\rfloor = 1 \rightarrow$ Code can correct only 1 error

Standard array for the C(5,2) code is

00000	10111	01101	11010
00001	10110	01100	11011
00010	10101	01111	11000
00100	10011	01001	11110
01000	11111	00101	10010
10000	00111	11101	01010
10001	00110	11100	01011
10100	00011	11001	01110

weight 1 errors

weight-2 errors

$2^{5-2} = 8$ rows

$\rightarrow$ our code cannot correct errors below this line

$2^k = 2^2 = 4$ colums.

$4 \times 8 = 32 = 2^5$ words in the standard array

(56)

Assume that we wanna transmit dataword

$\bar{d} = 11$

which is encoded as

$$\bar{c} = \bar{d}\,G \longrightarrow \bar{c} = (11)\begin{pmatrix} 10 & 111 \\ 01 & 101 \end{pmatrix}$$

$$= (11010)$$

$\bar{c} = (11010)$ is transmitted

and a single error occured

let the error word be $\bar{e} = (01000)$

The received word is $\bar{r} = \bar{c} + \bar{e}$

$$\bar{r} = 10010$$

we locate the received word $\bar{r}$ in the standard array and decide on the error pattern (coset leader)

which is $e = (01000)$

the received word is decoded as

$$r' = r + e \longrightarrow r' = (10010) + (01000)$$

$$= (11010)$$

and the data word which generates code word (11010) is decoded as 11.

(57) Properties of cosets:

If CS_1 & CS_2 are two cosets

then $CS_1 \cap CS_2 = \phi$

i.e, cosets do not contain common elements

$CS_1 \cup CS_2 \cup \ldots CS_{2^{n-k}} = V_n$

$V_n \rightarrow$ Vector space of n-tuples

$V_n \rightarrow$ Contains 2^n elements (tuples)

Syndrome Table:

Standard array may be difficult to construct for some codes. Instead syndrome table is used for its easiness.

$s = eH^T$ $\quad e \rightarrow$ error pattern

$H \rightarrow$ Parity check matrix of the code

Ex: Determine the syndrome table for the (6,3) single error-correcting code with the parity check matrix
$$H = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]_{3\times 6}$$

Sln:

$s = eH^T$

e	s
000001	001
000010	010
000100	100
001000	110
010000	101
100000	011

$\rightarrow$ Syndrome table.

Remark: $s = eH^T$ if $H = [h_1^T \; h_2^T \; \cdots \; h_{n-k}^T]$

Single error patterns produce

$$s_i = [0 \cdots 0 \; \underset{\substack{\downarrow \\ i\text{th} \\ \text{location}}}{1} \; 0 \cdots 0] \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_i \\ \vdots \\ h_{n-k} \end{bmatrix} \rightarrow s_i = h_i$$

(58)

In other words single error patterns produce syndromes which are colums of the parity check matrix. Double or more error patterns produce syndromes which are linear combinations of colums of the parity check matrix.

Error Detection & Correction

$r \rightarrow$ received word

(1) Calculate $s = rH^T$

(2) Obtain e corresponding to s as given in the syndrome table

(3) The decoder's estimate of the codeword is

$$\hat{c} = r + e$$

Remark: Syndrome table gives a concise representation of the standard array

Ex: Syndrome table for (6,3) linear code is given

e	s
000001	001
000010	010
000100	100
001000	110
010000	101
100000	011

Assume that $c(110110)$ is transmitted. Determine the decoder's estimate of c when c incurs the error pattern

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{3\times 6}$$

a (6,3) single error correcting code

(60)

a) $e=(001000)$ b) $e=(011011)$

c) $e=(000111)$

Sln:

a) $r=c+e \rightarrow r=(110110)+(001000)$

$r=(111110)$

$$S=rH^T \rightarrow S=(111110)\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{pmatrix} \rightarrow S=(110)$$

from syndrome table $s(110)$ gives $e=(001000)$ as the decoder's estimate of the error pattern. The decoder estimates the codeword to be

$$\hat{c}=r+e \rightarrow \hat{c}=(111110)+(00100)$$
$$=(110110)$$

which is the correct codeword.

b) $e=(011011) \rightarrow r=c+e$

$$=(110110)+(011011)$$
$$=(101101)$$

$$S=rH^T \rightarrow S=(101101)\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\\1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=(000)$$

Decoder assumes that no error occured which is incorrect.

(61)

c) $e=(000111)$ $r=c+e$
$=(110001)$

$s=eH^T \rightarrow s=(111)$ which is absent in syndrome table, we can say that error occurred but we don't know how to correct it.

$$H=\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$h_1 \quad h_2 \quad h_3 \quad h_4 \quad h_5 \quad h_6$

$s=eH^T$

$s=(111)$

corresponds to the sum of

$h_1+h_2+h_6$

which means that

$e=(110001)$

OR $s=(111)$
corresponds to the sum of
h_4, h_5, h_6 which means that
e can be $e=(000111)$

OR $s=(111)$
corresponds to the sum of
$h_2 \& h_5 \rightarrow e=(010010)$

So we cannot be sure about the error pattern occured.

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Shortened & Extended Linear Codes:

An (n,k) linear code can be shortened to a $(n-i, k-i)$ shortened linear code by setting the first i information bits to 0.

- The generator matrix of the shortened code is obtained by omitting the first i rows and colums of G and the parity-check matrix is obtained by omitting the first i columns of H.

To extend (n,k) to $(n+1,k)$ linear code add an overall parity check bit to produce $(n+1,k)$ code

$G \to$ generator matrix of (n,k)
$G' \to$ " " of $(n+1,k)$

G' is formed by adding a column to G which is obtained by summing all the other columns of G.

Ex: $G \to$ generator matrix of $(7,4)$ code

$$G = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

summ all of them

$$G' = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 & \to 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & \to 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & \to 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & \to 1 \end{array}\right]$$

The parity check matrix H' of the extended code $(n+1,k)$ is obtained as

$$H' = \left[\begin{array}{c|c} H & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 1\,1\,1 \cdots & 1\end{array}\right]$$

H is the parity check matrix of (n,k) code.

Ex:

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$H' = \left[\begin{array}{c|c} H & \bar{0} \\ \hline \bar{1} & \end{array}\right] \rightarrow H' = \left[\begin{array}{ccccccc|c} \multicolumn{7}{c|}{\overbrace{\hphantom{1\;1\;1\;0\;1\;0\;0}}^{H}} & \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$\rightarrow \bar{0}$

$\downarrow \bar{1}$

Exercise:

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{3\times 7} \rightarrow (7,3) \text{ code generator matrix}$$

a) Determine systematic form of G

b) Find the parity check matrix of the code.

c) $d_{min} = ?$ d) Error correction & detection capability

e) Syndrome table?

f) Extend $(7,3)$ to $(8,3)$ linear code.
find G' of $(8,3)$ code find H' of $(8,3)$ code.

(64) Exercises

$$H=\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

is the parity check matrix of (7,4) code.

a) Find the generator matrix in systematic form.

b) Find dmin using parity check matrix only

c) Error correcting capability t

d) Number of words in Hamming spheres with radius t

e) Verify that Singleton bound is satisfied

f) Verify that Hamming bound is satisfied

g) Construct syndrome table for this code

h) Find the extended (8,4) code parity-check & generator matrices

i) Find dmin of (8,4) code
error correction & detection capability?

j) Verify Singleton and Hamming bounds for (8,4) code.

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Some Important Codes:

Single parity check codes:

$c = (\underbrace{d_1 \dots d_k}_{d}\ p)$ $\quad p = d_1 + d_2 + \dots d_k$

Ex: $d = (1\,0\,1\,1)$ $\quad c = (1011\ \underbrace{1+0+1+1}_{1})$

$c = (1011\ 1)$

$c = (d_1\, d_2 \dots d_k\ p)$ $\quad p = 0$ if information bits contain an even number of 1's.

$p = 1$ otherwise

If odd parity is required then $p = (d_1 + d_2 + \dots d_k) + 1$

Decoding:

$r = (r_1 \dots r_n) \to$ received word.

$s = r_1 + r_2 + \dots r_n \to$ check sum

for even parity code

If $s = 0$ then r is a codeword

otherwise

The decoder is able to detect all single bit errors.

(66) Golay Codes

Discovered in the late 1940s.

Generator matrix of Golay code

$$G_{24} = (I_{12} \mid A)$$

I identity matrix of size 12×12

A is the 12×12 matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Golay code (G_{24}) is self dual

i.e, $G_{24} G_{24}^{T} = 0$

The weight of every codeword is a multiple of 4

i.e, Codewords Hamming weight 0, 8, 12, 16 or 24

The Golay code is an exactly three error correcting code.

(67) The Voyager 1 and 2 spacecrafts were launched towards Jupiter and Saturn in 1977. Golay code was used in the encoding and decoding of the general science and engineering (GSE) data for the missions.

Reed Muller Codes:

Generator matrix of Reed Muller code:

Reed Muller code $R(1,m) \leftarrow$ binary Reed Muller code.

$R(1,m) \longrightarrow (2^m, m+1)$

minimum distance of $R(1,m)$ is 2^{m-1}

Generator matrix of $R(1,1)$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Generator matrix G_m of $R(1,m)$ is

$$G_m = \begin{pmatrix} G_{m-1} & G_{m-1} \\ 0 \cdots 0 & 1 \cdots 1 \end{pmatrix}$$

Ex: Find generator matrices of $R(1,2)$ & $R(1,3)$ and compute minimum distances

Sln: $G_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $G_2 = \begin{pmatrix} G_1 & G_1 \\ 00 & 11 \end{pmatrix} \to G_2 = \begin{pmatrix} 11 & 11 \\ 01 & 01 \\ 00 & 11 \end{pmatrix}$

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$$G_2 = \begin{pmatrix} 11 & 11 \\ 01 & 01 \\ 00 & 11 \end{pmatrix} \rightarrow d_{min} = 2^{2-1} = 2$$

$$G_3 = \begin{pmatrix} G_2 & G_2 \\ \bar{0} & \bar{1} \end{pmatrix} \rightarrow G_3 = \begin{pmatrix} 11\,11 & 11\,11 \\ 01\,01 & 01\,01 \\ 00\,11 & 00\,11 \\ 0000 & 1111 \end{pmatrix}$$

d_{min} of $R(1,3)$ is $d_{min} = 2^{3-1} = 4$

Exercise: Find generator matrix of $R(1,4)$ and compute the minimum distance.

Hamming Codes:

Single error correcting block codes whose block lengths n and information lengths k satisfy

$n = 2^r - 1 \qquad k = 2^r - 1 - r \qquad$ OR $\qquad n = 2^r - 1$, $k = n - r$

where $r = n - k$

The columns of the parity check matrix (H) of Hamming code consists of all non-zero binary r-tuples.

(69)

Ex:

Let $r=3$ then

$n = 2^r - 1 \rightarrow n = 2^3 - 1$
$= 7$

$k = n - r \rightarrow k = 7 - 3 = 4$

Hence $(7,4) \rightarrow$ There exists such a Hamming Code.

$G = [I \mid P]_{k \times n} \rightarrow H = [-P^T \mid I]$

Parity check matrix of (7,4) Hamming code is formed using all binary 3-tuples as

$$H = \left[\begin{array}{cccc|ccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

columns can be permuted

same as in here but prefer to keep it as identity matrix since it is used while forming generator matrix G

The generator matrix of (7,4) Hamming code is obtained from parity check matrix H as

$$G = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]_{4 \times 7}$$

Remark: Colums of H (in Hamming code) consists of all non-zero r-tuples $r = n - k$.

Exercise: Obtain standard array and syndrome table for (7,4) Hamming code.

Example:

Systematic Parity-Check Matrix for (15,11) Hamming code.

$$H = \left[\begin{array}{ccccccccccc|cccc} 0&0&0&0&1&1&1&1&1&1&1&1&0&0&0 \\ 1&1&1&0&0&0&0&1&1&1&1&0&1&0&0 \\ 0&1&1&1&0&1&1&0&0&1&1&0&0&1&0 \\ 1&0&1&1&1&0&1&0&1&0&1&0&0&0&1 \end{array}\right]_{4\times 15}$$

$H = [-P^T \mid I] \to G = [I \mid P]$

$n = 15$

$k = 11$

$$G = \left[\begin{array}{ccccccccccc|c} 1&0&0&0&0&0&0&0&0&0&0&0101 \\ 0&1&0&0&0&0&0&0&0&0&0&0110 \\ 0&0&1&0&0&0&0&0&0&0&0&0111 \\ 0&0&0&1&0&0&0&0&0&0&0&0011 \\ 0&0&0&0&1&0&0&0&0&0&0&1001 \\ 0&0&0&0&0&1&0&0&0&0&0&1010 \\ 0&0&0&0&0&0&1&0&0&0&0&1011 \\ 0&0&0&0&0&0&0&1&0&0&0&1100 \\ 0&0&0&0&0&0&0&0&1&0&0&1101 \\ 0&0&0&0&0&0&0&0&0&1&0&1110 \\ 0&0&0&0&0&0&0&0&0&0&1&1111 \end{array}\right]_{\substack{k\times n \\ \downarrow\ \ \downarrow \\ 11\ \ 15}}$$

Extended Hamming Codes:

If H_r is the parity check matrix of Hamming code $(2^r - 1, 2^r - 1 - r)$

Parity check matrix of the extended Hamming code H_r is $H_r' = \left[\begin{array}{c|c} H_r & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ \hline 1 \cdots 1 & 1 \end{array}\right]$

(71)

- The dual of the binary Hamming code is called a binary simplex code

- The dual code of $R(1,m)$ is equivalent to the extended binary Hamming code.
Reed Muller Code

i.e, if Hamming code is $(\underline{2^r-1}, 2^r-1-r)$

its extended is $(\underline{2^r}, 2^r-1-r)$

Decoding Hamming Codes:

Consider (7,4) Hamming Code:

$$H=\left[\begin{array}{cccc|ccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]_{3\times 7} \rightarrow G=\left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

Syndrome table is of

$s = eH^T$

e	s
0000001	001
0000010	010
0000100	100
0001000	111
0010000	110
0100000	101
1000000	011

Consider data word $d = 1000 \rightarrow c = dG \rightarrow c = 1000011$

Assume 1-bit error occured $e = 1000000$

$r = c+e \rightarrow r = 0000011 \rightarrow s = re^H$

$\rightarrow s = 011$

which corresponds to $e = 1000000$

Now consider that the tuples in $\mathcal{H}$ are in increasing order

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

which corresponds to the generator matrix

$$G = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

OR $$G = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Remark:</u> To obtain G from H (not in systematic form) put H into systematic form and form G. Then permute H to get its non systematic form. Do the same permutations on G to obtain its non-systematic form.

<u>Ex:</u>

$$H_s = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]_{2\times 4} \rightarrow G_s = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]_{2\times 4}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$H_{n\text{-}s} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \rightarrow G_{ns} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$G_{ns} H_{ns}^T \overset{?}{=} 0$ yes ✓

(73) Continued Hamming Code

H is in non-systematic form, increasing duples

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}_{3\times 7} \rightarrow \text{OR can be } H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

$$G = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{4\times 7}$$

consider data-word $d = 1000$

$c = dG \longrightarrow c = 0101010$1

assume error $e = 1000000$

$r = c + e \longrightarrow r = 1101010$1

$s = rH^T \rightarrow s = \underbrace{100}_{1}$

shows the location of error.

Syndrome table:

e	s	
1000000	100	→ 1
0100000	010	→ 2
0010000	110	→ 3
0001000	001	→ 4
0000100	101	→ 5
0000010	011	→ 6
0000001	111	→ 7

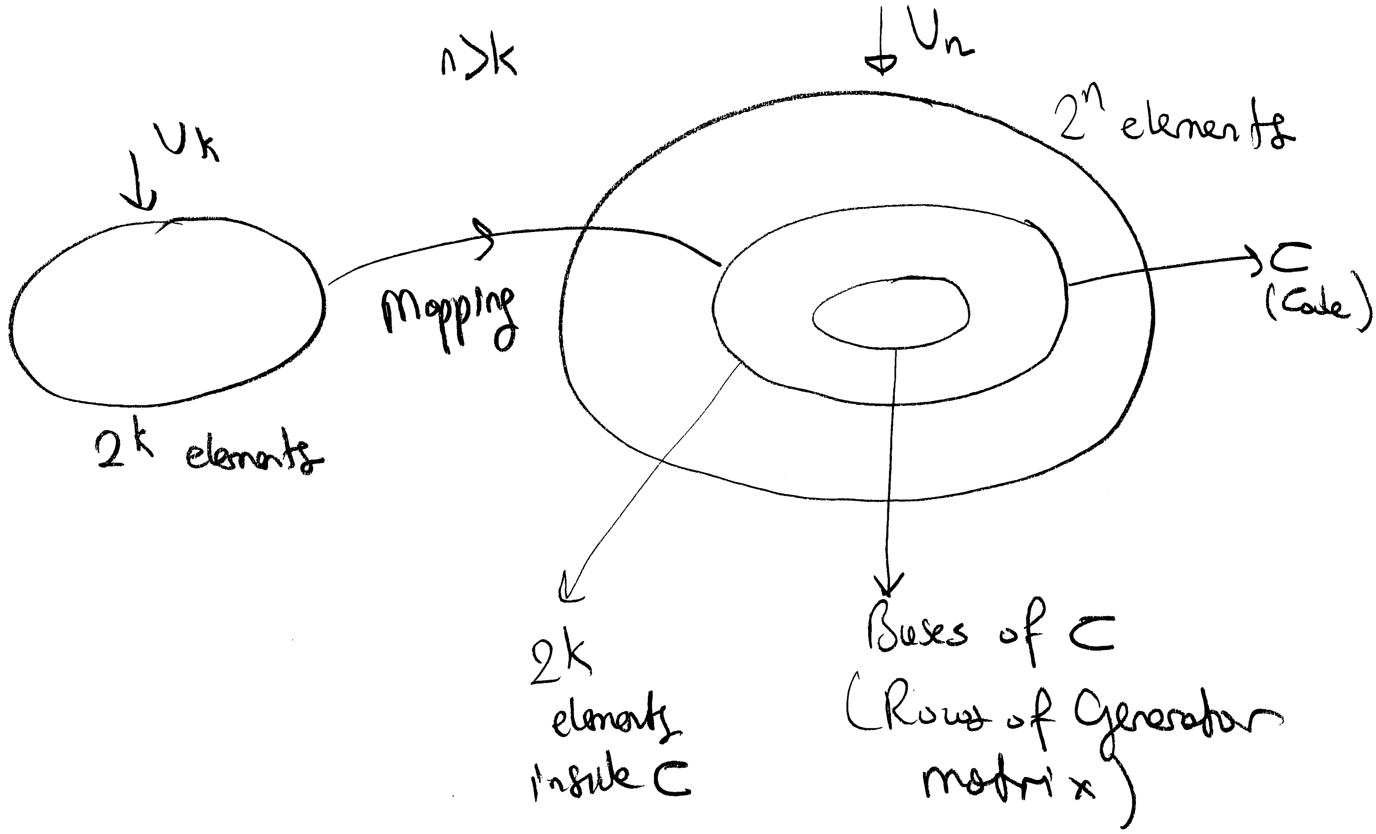
If we had used $H = \begin{bmatrix} 0 & \cdots & 1 \\ 0 & & 1 \\ 1 & \cdots & 1 \end{bmatrix}$

then

e	s	
1000000	111	→ 7
0100000	011	→ 6
0010000	101	→ 5
0001000	001	→ 4
0000100	110	→ 3
0000010	010	→ 2
0000001	100	→ 1

shows the location of error

Linear Block Codes (Summary)



The distance between codewords (block of bits) in C is larger than in U_k, therefore using C instead of U_k (they have the same number of elements) makes it more difficult to make errors.

But C requires to transmit more bits on the channel for the same information sequence, and therefore more bandwidth is needed.